CHAPTER

7

EXPONENTIAL FUNCTIONS

Chapter Table of Contents

7-1 Laws of Exponents

7-2 Zero and Negative Exponents

7-3 Fractional Exponents

7-4 Exponential Functions and Their Graphs

7-5 Solving Equations Involving Exponents

7-6 Solving Exponential Equations

7-7 Applications of Exponential Functions

Chapter Summary

Vocabulary

Review Exercises

Cumulative Review

The use of exponents to indicate the product of equal factors evolved through many different notations. Here are some early methods of expressing a power using an exponent. In many cases, the variable was not expressed. Each is an example of how the polynomial $9x^4 + 10x^3 + 3x^2 + 7x + 4$ was written.

- Joost Bürgi (1552–1632) used Roman numerals above the coefficient to indicate an exponent:

$$\overset{\text{IV}}{9} + \overset{\text{III}}{10} + \overset{\text{II}}{3} + \overset{\text{I}}{7} + 4$$

- Adriaan van Roomen (1561–1615) used parentheses:

$$9(4) + 10(3) + 3(2) + 7(1) + 4$$

- Pierre Hérigone (1580–1643) placed the coefficient before the variable and the exponent after:

$$9x4 + 10x3 + 3x2 + 7x1 + 4$$

- James Hume used a raised exponent written as a Roman numeral:

$$9x^{\text{iv}} + 10x^{\text{iii}} + 3x^{\text{ii}} + 7x^{\text{i}} + 4$$

- René Descartes (1596–1650) introduced the symbolism that is in common use today:

$$9x^4 + 10x^3 + 3x^3 + 7x + 4$$

7-1 LAWS OF EXPONENTS

When multiplying or dividing algebraic terms, we use the rules for exponents when the terms involve powers of a variable. For example:

$$a^3b^2(3a^2b) = 3a^5b^3$$
$$x^4y^5 \div x^3y^2 = xy^3$$
$$(c^3d^2)^2 = c^6d^4$$

These examples illustrate the following rules for exponents. If a and b are positive integers, then:

Multiplication: $x^a \cdot x^b = x^{a+b}$

Division: $x^a \div x^b = x^{a-b}$ or $\frac{x^a}{x^b} = x^{a-b} \quad (x \neq 0)$

Raising to a Power: $(x^a)^b = x^{ab}$

Power of a Product: $(xy)^a = x^a \cdot y^a$

Power of a Quotient: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

In the first example given above, we need the commutative and associative laws to group powers of the same base in order to apply the rules for exponents.

$$\begin{aligned} a^3b^2(3a^2b) &= 3a^2a^3b^2b && \text{Commutative property} \\ &= 3(a^2a^3)(b^2b) && \text{Associative property} \\ &= 3a^{2+3}b^{2+1} && x^a \cdot x^b = x^{a+b} \\ &= 3a^5b^3 \end{aligned}$$

Note: When a variable has no exponent, the exponent is understood to be 1.

In the second example, we use the rule for the division of powers with like bases:

$$\begin{aligned} x^4y^5 \div x^3y^2 &= \frac{x^4y^5}{x^3y^2} \\ &= \frac{x^4}{x^3} \cdot \frac{y^5}{y^2} \\ &= x^{4-3}y^{5-2} && \frac{x^a}{x^b} = x^{a-b} \\ &= xy^3 \end{aligned}$$

In the third example, we can use the rule for multiplying powers with like bases or the rule for raising a power to a power:

$$\begin{aligned} &(c^3d^2)^2 \\ &= (c^3d^2)(c^3d^2) && x^2 = x \cdot x \\ &= c^3c^3d^2d^2 && \text{Commutative property} \\ &= c^{3+3}d^{2+2} && x^a \cdot x^b = x^{a+b} \\ &= c^6d^4 \end{aligned}$$

$$\begin{aligned} &(c^3d^2)^2 \\ &= (c^3)^2(d^2)^2 && (xy)^a = x^a \cdot y^a \\ &= c^{3(2)}d^{2(2)} && (x^a)^b = x^{ab} \\ &= c^6d^4 \end{aligned}$$

EXAMPLE 1

Simplify: $\frac{a^4(3b)^7}{a(3b)^2}$

Solution $\frac{a^4(3b)^7}{a(3b)^2} = \frac{a^4}{a^1} \cdot \frac{(3b)^7}{(3b)^2} = a^{4-1}(3b)^{7-2} = a^3(3b)^5 = 3^5a^3b^5 = 243a^3b^5$ *Answer*

Alternative Solution $\frac{a^4(3b)^7}{a(3b)^2} = \frac{a^4(3)^7(b)^7}{a^1(3)^2(b)^2} = a^{4-1}(3)^{7-2}(b)^{7-2} = a^3(3)^5(b)^5 = 243a^3b^5$ *Answer*

EXAMPLE 2

What is the value of the fraction $\frac{(-3)^4(2)^5}{(-3(2))^3}$?

Solution $\frac{(-3)^4(2)^5}{(-3(2))^3} = \frac{(-3)^4(2)^5}{(-3)^3(2)^3} = (-3)^{4-3}(2)^{5-3} = (-3)^1(2)^2 = (-3)(4) = -12$ *Answer*

Evaluating Powers

Note that the expression $(3a)^4$ is *not* equivalent to $3a^4$. We can see this by using the rule for multiplying powers with like bases or the rule for raising a power to a power. Let $a = 2$ or any number not equal to 0 or 1:

$$3a^4 = 3(a)(a)(a)(a) = 3(2)(2)(2)(2) = 48$$

$$(3a)^4 = (3a)(3a)(3a)(3a) = (3 \times 2)(3 \times 2)(3 \times 2)(3 \times 2) = (6)(6)(6)(6) = 1{,}296$$

or

$$(3a)^4 = (3)^4(a)^4 = (3)^4(2)^4 = (81)(16) = 1{,}296$$

Similarly, since $-a = -1a$, the expression $-a^4$ is *not* equivalent to $(-a)^4$. Let $a = 2$ (or any number not equal to 0 or 1):

$$-a^4 = -(a)(a)(a)(a) = -(2)(2)(2)(2) = -16$$

$$(-a)^4 = (-a)(-a)(-a)(-a) = (-2)(-2)(-2)(-2) = 16$$

Exercises

Writing About Mathematics

1. Randy said that $(2)^3(5)^2 = (10)^5$. Do you agree with Randy? Justify your answer.

2. Natasha said that $(2)^3(5)^3 = (10)^3$. Do you agree with Natasha? Justify your answer.

Developing Skills

In 3–26, simplify each expression. In each exercise, all variables are positive.

3. $x^3 \cdot x^4$
4. $y \cdot y^5$
5. $x^6 \div x^2$
6. $y^4 \div y$
7. $(x^5)^2$
8. $(2y^4)^3$
9. $10^2 \cdot 10^4$
10. $-2^6 \cdot 2^2$
11. $x^4 \cdot x^2y^3$
12. $xy^5 \cdot xy^2$
13. $-(3x^3)^2$
14. $(-3x^3)^2$
15. $x^8y^6 \div (x^3y^5)$
16. $x^9y^7 \div (x^8y^7)$
17. $(x^2y^3)^3 \cdot (x^2y)$
18. $(-2x)^4 \cdot (2x^3)^2$
19. $\frac{(4x)^3}{4x^3}$
20. $\frac{3(x^3)^4y^5}{3x^7}$
21. $\frac{-x^4y^6}{(-x^3y^4)}$
22. $\left(\frac{x^3y^5}{(xy^2)^2}\right)^2$
23. $\frac{x^2(y^3z)^3}{(x^2y)^2z}$
24. $\left(\frac{2a^3}{a^2}\right)^5 \cdot b$
25. $\frac{4(ab)^2c^5}{abc}$
26. $\frac{(a^x)^yb}{a^{xy}}$

Applying Skills

27. What is the value of n if $8^3 = 2^n$?
28. What is the value of a if $27^2 = 9^a$?
29. If $3^{a+1} = x$ and $3^a = y$, express y in terms of x.
30. If $25^{b+1} = x$ and $5^b = y$, express x in terms of y.

In 31–33, the formula $A = P(1 + r)^t$ expresses the amount A to which P dollars will increase if invested for t years at a rate of r per year.

31. Find A when $P = \$500$, $r = 0.04$ and $t = 5$.
32. Find the amount to which \$2,400 will increase when invested at 5% for 10 years.
33. What is the minimum number of years that \$1 must be in invested at 5% to increase to \$2? (Use a calculator to try possible values of t.)

7-2 ZERO AND NEGATIVE EXPONENTS

Zero Exponents

We know that any nonzero number divided by itself is 1:

$$\frac{4}{4} = 1 \qquad \frac{3^4}{3^4} = \frac{81}{81} = 1 \qquad \frac{7^5}{7^5} = 1$$

In general, for $x \neq 0$ and n a positive integer:

$$\frac{x^n}{x^n} = 1$$

Can we apply the rule for the division of powers with like bases to these examples?

$$\frac{4}{4} = \frac{2^2}{2^2} = 2^{2-2} = 2^0 \qquad \frac{3^4}{3^4} = 3^{4-4} = 3^0 \qquad \frac{7^5}{7^5} = 7^{5-5} = 7^0$$

In general, for $x \neq 0$ and n a positive integer:

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

In order for the rule for the division of powers with like bases to be consistent with ordinary division, we must be able to show that for all $x \neq 0, x^0 = 1$.

Multiplication:	$x^n \cdot x^0 = x^{n+0} = x^n$	$x^n \cdot x^0 = x^n \cdot 1 = x^n$
Division:	$x^n \div x^0 = x^{n-0} = x^n$	$x^n \div x^0 = x^n \div 1 = x^n$
Raising to a Power:	$(x^0)^n = x^{0 \cdot n} = x^0 = 1$	$(x^0)^n = 1^n = 1$
Power of a Product:	$(xy)^0 = x^0 \cdot y^0 = 1 \cdot 1 = 1$	$(xy)^0 = 1$
Power of a Quotient:	$\left(\frac{x}{y}\right)^0 = \frac{x^0}{y^0} = \frac{1}{1} = 1$	$\left(\frac{x}{y}\right)^0 = 1$

Therefore, it is reasonable to make the following definition:

DEFINITION

If $x \neq 0, x^0 = 1$.

EXAMPLE 1

Find the value of $\frac{3x^2}{x^2}$ in two different ways.

Solution $\frac{3x^2}{x^2} = 3 \cdot \frac{x^2}{x^2} = 3(1) = 3$ or $\frac{3x^2}{x^2} = 3x^{2-2} = 3x^0 = 3(1) = 3$

Answer 3

Negative Exponents

When using the rule for the division of powers with like bases, we will allow the exponent of the denominator to be larger than the exponent of the numerator.

$$\frac{3^4}{3^6} = \frac{1 \times 3^4}{3^2 \times 3^4} = \frac{1}{3^2} \times \frac{3^4}{3^4} = \frac{1}{3^2} \times 1 = \frac{1}{3^2} \qquad \frac{3^4}{3^6} = 3^{4-6} = 3^{-2}$$

$$\frac{5}{5^4} = \frac{1 \times 5}{5^3 \times 5} = \frac{1}{5^3} \times \frac{5}{5} = \frac{1}{5^3} \times 1 = \frac{1}{5^3} \qquad \frac{5}{5^4} = 5^{1-4} = 5^{-3}$$

$$\frac{x^a}{x^{a+b}} = \frac{1 \cdot x^a}{x^b \cdot x^a} = \frac{1}{x^b} \cdot \frac{x^a}{x^a} = \frac{1}{x^b} \cdot 1 = \frac{1}{x^b} \qquad \frac{x^a}{x^{a+b}} = x^{a-(a+b)} = x^{-b}$$

This last example suggests that $x^{-b} = \frac{1}{x^b}$. Before we accept that $x^{-b} = \frac{1}{x^b}$, we must show that this is consistent with the rules for powers.

Multiplication: $2^5 \cdot 2^{-3} = 2^{5+(-3)} = 2^2$

$2^5 \cdot 2^{-3} = 2^5 \cdot \frac{1}{2^3} = 2^2$

Division: $6^2 \div 6^{-5} = 6^{2-(-5)} = 6^7$

$6^2 \div 6^{-5} = 6^2 \div \frac{1}{6^5} = 6^2 \cdot 6^5 = 6^7$

Raising to a Power: $(3^{-4})^{-2} = 3^{-4(-2)} = 3^8$

$(3^{-4})^{-2} = \left(\frac{1}{3^4}\right)^{-2} = \frac{1}{3^{-8}} = \frac{1}{\frac{1}{3^8}} = \frac{3^8}{1} = 3^8$

Power of a Product: $(x^2y^{-3})^4 = x^8y^{-12} = x^8 \cdot \frac{1}{y^{12}} = \frac{x^8}{y^{12}}$

$(x^2y^{-3})^4 = (x^2 \cdot \frac{1}{y^3})^4 = (x^2)^4\left(\frac{1}{y^3}\right)^4 = \frac{x^8}{y^{12}}$

Power of a Quotient: $\left(\frac{x^3}{y^5}\right)^{-2} = \frac{x^{-6}}{y^{-10}} = \frac{\frac{1}{x^6}}{\frac{1}{y^{10}}} = \frac{y^{10}}{x^6}$

$\left(\frac{x^3}{y^5}\right)^{-2} = \frac{1}{\left(\frac{x^3}{y^5}\right)^2} = \frac{1}{\frac{x^6}{y^{10}}} = \frac{y^{10}}{x^6}$

Therefore, it is reasonable to make the following definition:

DEFINITION

If $x \neq 0, x^{-n} = \frac{1}{x^n}$.

EXAMPLE 2

Show that for $x \neq 0, \frac{1}{x^{-n}} = x^n$.

Solution

$$\frac{1}{x^{-n}} = \frac{1}{\frac{1}{x^n}} = \frac{1}{\frac{1}{x^n}} \cdot \frac{x^n}{x^n} = \frac{x^n}{1} = x^n$$

EXAMPLE 3

Write $\frac{a^4b^{-3}}{ab^{-2}}$ with only positive exponents.

Solution $\frac{a^4b^{-3}}{ab^{-2}} = a^{4-1}b^{-3-(-2)} = a^3b^{-1} = a^3 \cdot \frac{1}{b} = \frac{a^3}{b}$ *Answer*

EXAMPLE 4

Express as a fraction the value of $2x^0 + 3x^{-3}$ for $x = 5$.

Solution $2x^0 + 3x^{-3} = 2(5)^0 + 3(5)^{-3} = 2(1) + 3 \times \frac{1}{5^3} = 2 + \frac{3}{125} = \frac{250}{125} + \frac{3}{125} = \frac{253}{125}$

Answer $\frac{253}{125}$

Exercises

Writing About Mathematics

1. Kim said that $a^0 + a^0 = a^{0+0} = a^0 = 1$. Do you agree with Kim? Explain why or why not.

2. Tony said that $a^0 + a^0 = 2a^0 = 2$. Do you agree with Tony? Explain why or why not?

Developing Skills

In 3–10, write each expression as a rational number without an exponent.

3. 5^{-1} **4.** 4^{-2} **5.** 6^{-2} **6.** $\left(\frac{1}{2}\right)^{-1}$

7. $\left(\frac{1}{5}\right)^{-3}$ **8.** $\left(\frac{2}{3}\right)^{-1}$ **9.** $\frac{3^0}{4^{-2}}$ **10.** $\frac{(2 \cdot 5)^{-4}}{5^{-2}}$

In 11–22, find the value of each expression when $x \neq 0$.

11. 7^0 **12.** $(-5)^0$ **13.** x^0 **14.** -4^0

15. $(4x)^0$ **16.** $4x^0$ **17.** $-2x^0$ **18.** $(-2x)^0$

19. $\left(\frac{3}{4}\right)^0$ **20.** $\frac{3^0}{4}$ **21.** $\frac{3^0}{4^0}$ **22.** $\frac{3x^0}{(4x)^0}$

In 23–34, evaluate each function for the given value. Be sure to show your work.

23. $f(x) = x^{-3} \cdot x^4$; $f(1)$

24. $f(x) = x + x^{-5}$; $f(3)$

25. $f(x) = (2x)^{-6} \div x^3$; $f(-3)$

26. $f(x) = (x^{-7})^4$; $f(-6)$

27. $f(x) = \left(\frac{1}{x} + \frac{3}{2}\right)^{-2}$; $f(2)$

28. $f(x) = 10^x + 10^{-2x}$; $f(3)$

29. $f(x) = x^{-7} \div x^8$; $f\left(\frac{3}{4}\right)$

30. $f(x) = (3x^{-3} - 2x^{-3})^2$; $f(-2)$

31. $f(x) = x^8\left(x^{-2} + \frac{1}{x^3}\right)$; $f\left(\frac{1}{2}\right)$

32. $f(x) = \left(\frac{x^{-1}}{(2x)^{-2}}\right)^{-1}$; $f(8)$

33. $f(x) = \frac{1}{1 + \frac{2}{x^{-1}}}$; $f(-5)$

34. $f(x) = 4\left(\frac{1}{2}\right)^{-x} + 3\left(\frac{1}{2}\right)^{-x}$; $f(3)$

In 35–63, write each expression with only positive exponents and express the answer in simplest form. The variables are not equal to zero.

35. x^{-4} **36.** a^{-6} **37.** y^{-5} **38.** $2x^{-2}$

39. $7a^{-4}$ **40.** $-5y^{-8}$ **41.** $(2x)^{-2}$ **42.** $(3a)^{-4}$

43. $(4y)^{-3}$ **44.** $-(2x)^{-2}$ **45.** $-(3a)^{-4}$ **46.** $(-2x)^{-2}$

47. $\frac{1}{x^{-3}}$ **48.** $\frac{1}{y^{-7}}$ **49.** $\frac{3}{a^{-5}}$ **50.** $\frac{6}{a^{-4}}$

51. $\frac{9x^2}{a^{-3}}$ **52.** $\frac{(-x)^{-5}}{x^{-3}}$ **53.** $\frac{(2a)^{-1}}{2(a)^{-2}}$ **54.** $\frac{4y^{-3}}{2y^{-1}}$

55. $(xy^5z^{-2})^{-1}$

56. $(a^5b^{-5}c^{-4})^{-3}$

57. $\left(\frac{3m^{-3}}{2n^{-2}}\right)^{-3}$

58. $\left(\frac{6ab^4}{3x^{-3}y^{-4}}\right)^{-1}$

59. $\frac{x^5y^{-4}}{x^{-4}y^{-2}}$

60. $(3ab^{-2})(3a^{-1}b^3)$

61. $\frac{-64x^4a^{-2}}{2x^5b^{-4}}$

62. $\left(\frac{-49u^3v^4}{-7u^4v^7}\right)^{-1}$

63. $x^{-1} + x^{-5}$

In 64–75, write each quotient as a product without a denominator. The variables are not equal to zero.

64. $(xy) \div (xy^3)$

65. $(a^2b^3) \div (ab^5)$

66. $x^3 \div (x^3y^4)$

67. $12ab \div 2ab^2$

68. $\frac{1}{a^{-3}}$

69. $\frac{3}{x^4}$

70. $\frac{8}{4a^3}$

71. $\frac{36}{9x^{-5}}$

72. $\frac{3a^0b^{-3}}{a^{-1}b^{-3}}$

73. $\frac{20x^0y^{-5}}{4x^{-1}y^5}$

74. $\frac{15x^{-2}y^2}{3xy^5}$

75. $\frac{25a^5b^{-3}}{5^0a^{-1}b}$

76. Find the value of $a^0 + (4a)^{-1} + 4a^{-2}$ if $a = 2$.

77. Find the value of $(-5a)^0 - 5a^{-2}$ if $a = 3$.

78. If $\left(\frac{3}{4}\right)^{-3} + \left(\frac{8}{3}\right)^2 = \frac{2^a}{3^b}$, find the values of a and of b.

79. Show that $3 \times 10^{-2} = \frac{3}{100}$.

7-3 FRACTIONAL EXPONENTS

We know the following:

Since $3^2 = 9$: $\sqrt{9} = 3$ or $\sqrt{3^2}$

Since $5^3 = 125$: $\sqrt[3]{125} = 5$ or $\sqrt[3]{5^3}$

Since $2^4 = 16$: $\sqrt[4]{16} = 2$ or $\sqrt[4]{2^4}$

In general, for $x > 0$: $\sqrt[n]{x^n} = x$

In other words, raising a positive number to the nth power and taking the nth root of the power are inverse operations. Is it possible to express the nth root as a power? Consider the following example:

(1) Let x be any positive real number: $\sqrt{x} \cdot \sqrt{x} = x$

(2) Assume that $\sqrt{x}$ can be expressed as x^a: $x^a \cdot x^a = x$

(3) Use the rule for multiplying powers with like bases:

$$x^{a+a} = x^1$$
$$x^{2a} = x^1$$
$$2a = 1$$
$$a = \tfrac{1}{2}$$

(4) Therefore, we can write the following equality: $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x$

(5) Since for $x > 0$, $\sqrt{x}$ is one of the two equal factors of x, then: $\sqrt{x} = x^{\frac{1}{2}}$

We can apply this reasoning to any power and the corresponding root:

$$\underbrace{\sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \cdots \cdot \sqrt[n]{x}}_{n \text{ factors}} = x$$

Assume that $\sqrt[n]{x}$ can be expressed as x^a:

$$\underbrace{x^a \cdot x^a \cdot x^a \cdot x^a \cdot \cdots \cdot x^a}_{n \text{ factors}} = x$$

$$(x^a)^n = x^1$$
$$an = 1$$
$$a = \tfrac{1}{n}$$

For n a positive integer, $\sqrt[n]{x}$ is one of the n equal factors of x, and $x^{\frac{1}{n}}$ is one of the n equal factors of x. Therefore:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

Before we accept this relationship, we must verify that operations with radicals and with fractional exponents give the same results.

Multiplication: $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

$$\begin{aligned}(8)^{\frac{1}{2}} \times (2)^{\frac{1}{2}} &= (2^3)^{\frac{1}{2}} \times (2)^{\frac{1}{2}} \\ &= (2)^{\frac{3}{2}} \times (2)^{\frac{1}{2}} \\ &= (2)^{\frac{3}{2}+\frac{1}{2}} \\ &= (2)^{\frac{4}{2}} \\ &= (2)^2 = 4\end{aligned}$$

Division: $\frac{5}{\sqrt{5}} = \frac{5}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$

$$\frac{5^1}{5^{\frac{1}{2}}} = 5^{1-\frac{1}{2}} = 5^{\frac{1}{2}}$$

Raising to a Power: $\sqrt[3]{2^6} = \sqrt[3]{64} = 4$

$$(2^6)^{\frac{1}{3}} = 2^{6\left(\frac{1}{3}\right)} = 2^2 = 4$$

Verifying that the laws for power of a product and power of a quotient also hold is left to the student. (See Exercise 83.) These examples indicate that it is reasonable to make the following definition:

DEFINITION

If $x \geq 0$ and n is a positive integer, $\sqrt[n]{x} = x^{\frac{1}{n}}$.

From this definition and the rules for finding the power of a power and the definition of negative powers, we can also write the following:

$$\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m = x^{\frac{m}{n}} \qquad \frac{1}{\sqrt[n]{x^m}} = \frac{1}{\left(\sqrt[n]{x}\right)^m} = x^{-\frac{m}{n}}$$

Note: When $x < 0$, these formulas may lead to invalid results.

For instance: $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$ and $\frac{1}{3} = \frac{2}{6}$

However: $(-8)^{\frac{2}{6}} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$

Thus, $(-8)^{\frac{1}{3}} \neq (-8)^{\frac{2}{6}}$ even though the exponents are equal.

A calculator will evaluate powers with fractional exponents. For instance, to find $(125)^{\frac{2}{3}}$:

ENTER: 125 [^] [(] 2 [÷] 3 [)] [ENTER]

DISPLAY:

```
125^(2/3)
                    25
```

We can also evaluate $(125)^{\frac{2}{3}}$ by writing it as $\sqrt[3]{125^2}$ or as $\left(\sqrt[3]{125}\right)^2$.

ENTER: [MATH] [4] 125 [x²] [)] [ENTER]

DISPLAY:

```
³√(125²)
                    25
```

ENTER: [(] [MATH] [4] 125 [)] [)] [x²] [ENTER]

DISPLAY:

```
(³√(125))²
                    25
```

The calculator shows that $(125)^{\frac{2}{3}} = \sqrt[3]{125^2} = \left(\sqrt[3]{125}\right)^2$.

EXAMPLE 1

Find the value of $81^{-\frac{3}{4}}$.

Solution

$$81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{(3)^3} = \frac{1}{27} \quad \textit{Answer}$$

Calculator Solution

ENTER: 81 [^] [(-)] [(] 3 [÷] 4 [)] [MATH] [ENTER] [ENTER]

DISPLAY:

```
81^-(3/4)►Frac
            1/27
```

Note that we used the Frac function from the [MATH] menu to convert the answer to a fraction.

EXAMPLE 2

Find the value of $2a^0 - (2a)^0 + a^{\frac{1}{3}}$ if $a = 64$.

Solution

$$\begin{aligned} 2a^0 - (2a)^0 + a^{\frac{1}{3}} &= 2(64)^0 - [2(64)]^0 + 64^{\frac{1}{3}} \\ &= 2(1) - 1 + \sqrt[3]{64} \\ &= 2 - 1 + 4 \\ &= 5 \quad \textit{Answer} \end{aligned}$$

Exercises

Writing About Mathematics

1. Use exponents to show that for $a > 0$, $\left(\sqrt[n]{a}\right)^0 = 1$.

2. Use exponents to show that for $a > 0$, $\sqrt{\sqrt{a}} = \sqrt[4]{a}$.

Developing Skills

In 3–37, express each power as a rational number in simplest form.

3. $4^{\frac{1}{2}}$ **4.** $9^{\frac{1}{2}}$ **5.** $100^{\frac{1}{2}}$ **6.** $8^{\frac{1}{3}}$

7. $125^{\frac{1}{3}}$ **8.** $216^{\frac{1}{3}}$ **9.** $32^{\frac{1}{5}}$ **10.** $(3 \times 12)^{\frac{1}{2}}$

11. $(2 \times 8)^{\frac{1}{4}}$ **12.** $5(81)^{\frac{1}{4}}$ **13.** $-4(1{,}000)^{\frac{1}{3}}$ **14.** $49^{\frac{3}{2}}$

15. $8^{\frac{5}{3}}$ **16.** $27^{\frac{4}{3}}$ **17.** $10{,}000^{\frac{3}{4}}$ **18.** $32^{\frac{4}{5}}$

19. $9^{-\frac{1}{2}}$ **20.** $8^{-\frac{1}{3}}$ **21.** $100^{-\frac{3}{2}}$ **22.** $125^{-\frac{2}{3}}$

23. $3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$ **24.** $5^{\frac{1}{3}} \times 5^{\frac{2}{3}}$ **25.** $7^{\frac{3}{4}} \times 7^{\frac{5}{4}}$ **26.** $4 \times 4^{\frac{1}{2}}$

27. $32 \times 32^{\frac{1}{5}}$ **28.** $2^{\frac{1}{4}} \times 8^{\frac{1}{4}}$ **29.** $12^{\frac{5}{3}} \div 12^{\frac{2}{3}}$ **30.** $3^{\frac{7}{3}} \div 3^{\frac{1}{3}}$

31. $4^{\frac{2}{3}} \div 4^{\frac{1}{6}}$ **32.** $125^{\frac{2}{3}} \div 125^{\frac{1}{3}}$ **33.** $4^0 + 4^{-\frac{1}{2}}$ **34.** $9^{-2} + 9^{\frac{1}{2}}$

35. $2[(3)^{-2} + (4)^{-2}]^{-\frac{1}{2}}$ **36.** $\left(2.3 \times 10^{-\frac{1}{3}}\right)\left(5.2 \times 10^{-\frac{2}{3}}\right)$ **37.** $\dfrac{\left(2(3)^2 + \frac{1}{3^{-2}}\right)^{\frac{2}{3}}}{6\left(2 + \frac{1}{4}\right)^{-\frac{1}{2}}}$

In 38–57, write each radical expression as a power with positive exponents and express the answer in simplest form. The variables are positive numbers.

38. $\sqrt{7}$ **39.** $\sqrt{6}$ **40.** $\sqrt[3]{12}$ **41.** $\sqrt[3]{15}$

42. $\sqrt[4]{3}$ **43.** $\sqrt[5]{2^3}$ **44.** $(\sqrt[5]{9})^4$ **45.** $\dfrac{1}{(\sqrt{5})^3}$

46. $\sqrt{25a}$ **47.** $\sqrt{49x^2}$ **48.** $\sqrt{64a^3b^6}$ **49.** $\frac{1}{2}\sqrt{18a^6b^2}$

50. $\sqrt{9a^{-2}b^6}$ **51.** $\sqrt{\dfrac{3a}{4b}}$ **52.** $\sqrt[3]{27a^3}$ **53.** $\sqrt[4]{64x^5}$

54. $\dfrac{1}{\sqrt[5]{xyz^5}}$ **55.** $\sqrt{\dfrac{9a^{-2}}{4b^4}}$ **56.** $\sqrt[10]{\dfrac{w^{15}x^{20}}{y^5}}$ **57.** $\sqrt[8]{\sqrt[4]{a} \cdot \sqrt[4]{b^7}}$

In 58–73, write each power as a radical expression in simplest form. The variables are positive numbers.

58. $3^{\frac{1}{2}}$ **59.** $5^{\frac{1}{2}}$ **60.** $6^{\frac{1}{3}}$ **61.** $9^{\frac{1}{3}}$

62. $5^{\frac{3}{2}}$ **63.** $12^{\frac{5}{4}}$ **64.** $6^{\frac{5}{2}}$ **65.** $\dfrac{1}{5^{\frac{3}{2}}}$

66. $(x^{13})^{\frac{1}{7}}$ **67.** $(25x^2y)^{\frac{1}{2}}$ **68.** $(50ab^4)^{\frac{1}{2}}$ **69.** $(16a^5b^6)^{\frac{1}{4}}$

70. $\dfrac{(x^5y^6)^{\frac{1}{7}}}{z^{-\frac{3}{7}}}$ **71.** $\dfrac{5^{\frac{1}{3}}a^{\frac{2}{3}}}{4^{\frac{1}{3}}}$ **72.** $\left(\dfrac{-32x^{10}}{y^4}\right)^{\frac{1}{5}}$ **73.** $\dfrac{8^{\frac{1}{6}}a^{\frac{5}{6}}b^{\frac{3}{6}}}{(27c^4)^{\frac{1}{6}}}$

In 74–82, write each expression as a power with positive exponents in simplest form.

74. $\left(\dfrac{2a^{\frac{1}{2}}}{3a^{\frac{1}{6}}}\right)^6$ **75.** $\left(\dfrac{x^2y}{3x^4b^2}\right)^{\frac{2}{3}}$ **76.** $\left(\dfrac{4a^4b^6}{25a^{-1}b}\right)^{\frac{1}{2}}$

77. $\left(\dfrac{8a^2z^6}{27x^9a^{-4}z^{-1}}\right)^{\frac{1}{3}}$ **78.** $\sqrt{x^2y} \cdot \sqrt{x^4y^3}$ **79.** $\dfrac{\sqrt[6]{a^5}}{\sqrt[5]{a^5}}$

80. $\dfrac{\sqrt[3]{11x^5y^4}}{\sqrt{2x^5y^2}}$ **81.** $\dfrac{\sqrt[5]{48xy^2}}{\sqrt[3]{6x^2y^4}}$ **82.** $\left(\sqrt{2xy^2}\right)\left(\sqrt[4]{16x^2y}\right)$

83. Verify that the laws for power of a product and power of a quotient are true for the following examples. In each example, evaluate the left side using the rules for radicals and the right side using the rules for fractional exponents:

a. $(\sqrt[3]{27} \cdot \sqrt[3]{3})^2 \stackrel{?}{=} (27^{\frac{1}{3}} \cdot 3^{\frac{1}{3}})^2$ **b.** $\left(\dfrac{\sqrt{3}}{\sqrt{9}}\right)^3 \stackrel{?}{=} \left(\dfrac{3^{\frac{1}{2}}}{9^{\frac{1}{2}}}\right)^3$

7-4 EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

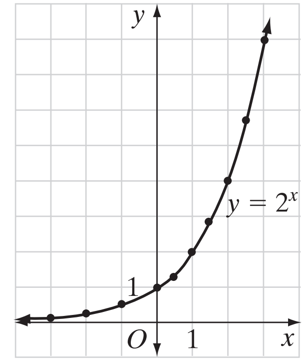
Two friends decide to start a chess club. At the first meeting, they are the only members of the club. To increase membership, they decide that at the next meeting, each will bring a friend. At the second meeting there are twice as many, or $2 \times 2 = 2^2$, members. Since they still want to increase membership, at the third meeting, each member brings a friend so that there are $2 \times 2^2 = 2^3$ members. If this pattern continues, then at the nth meeting, there will be 2^n members. We say that the membership has increased exponentially or is an example of **exponential growth**.

Often quantities that change exponentially decrease in size. For example, radioactive elements decrease by an amount that is proportional to the size of the sample. This is an example of **exponential decay**. The half-life of an element is the length of time required for the amount of a sample to decrease by one-half. If the weight of a sample is 1 gram and the half life is t years, then:

- After one period of t years, the amount present is $(1)\frac{1}{2} = 2^{-1}$ grams.
- After two periods of t years, the amount present is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = 2^{-2}$ grams.
- After three periods of t years, the amount present is $\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{8} = 2^{-3}$ grams.
- After n periods of t years, the amount present is $\frac{1}{2^n} = 2^{-n}$ grams.

Each of the examples given above suggests a function of the form $f(x) = 2^x$. To study the function $f(x) = 2^x$, choose values of x, find the corresponding value of $f(x)$, and plot the points. Draw a smooth curve through these points to represent the function.

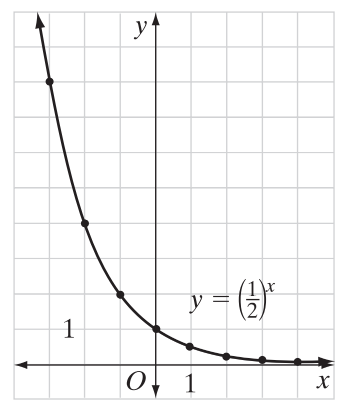
x	2^x	y	x	2^x	y
-3	$2^{-3} = \frac{1}{2^3}$	$\frac{1}{8}$	1	$2^1 = 2$	2
-2	$2^{-2} = \frac{1}{2^2}$	$\frac{1}{4}$	$\frac{3}{2}$	$2^{\frac{3}{2}} = \sqrt{2^3}$	$\sqrt{8} \approx 2.8$
-1	$2^{-1} = \frac{1}{2^1}$	$\frac{1}{2}$	2	$2^2 = 4$	4
0	$2^0 = 1$	1	$\frac{5}{2}$	$2^{\frac{5}{2}} = \sqrt{2^5}$	$\sqrt{32} \approx 5.7$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$	$\sqrt{2} \approx 1.4$	3	$2^3 = 8$	8



The function $y = 2^x$ is an increasing function. As we trace the graph from left to right, the values of y increase, that is, as x increases, y also increases.

If we reflect the graph of $y = 2^x$ over the y-axis, $(x, y) \rightarrow (-x, y)$ and the equation of the image is $y = 2^{-x}$, that is, $y = \frac{1}{2^x}$ or $y = \left(\frac{1}{2}\right)^x$. Compare the graph shown below with the values of $\left(\frac{1}{2}\right)^x$ shown in the table.

x	$\left(\frac{1}{2}\right)^x$	y	x	$\left(\frac{1}{2}\right)^x$	y
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3$	8	1	$\left(\frac{1}{2}\right)^1$	$\frac{1}{2}$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2$	4	2	$\left(\frac{1}{2}\right)^2$	$\frac{1}{4}$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1$	2	3	$\left(\frac{1}{2}\right)^3$	$\frac{1}{8}$
0	$\left(\frac{1}{2}\right)^0$	1	4	$\left(\frac{1}{2}\right)^4$	$\frac{1}{16}$



Note that the function $y = \left(\frac{1}{2}\right)^x$ is a decreasing function. As we trace the graph from left to right, the values of y decrease, that is, as x increases, y decreases.

An **exponential function** is a function of the form $y = b^x$ where b is a positive number not equal to 1. If $b > 1$, the function is an increasing function. If $0 < b < 1$, the function is a decreasing function.

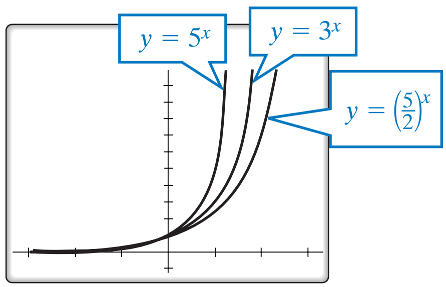
We can use a graphing calculator to draw several exponential functions with $b > 1$. For example, let $y = \left(\frac{5}{2}\right)^x$, $y = 3^x$, and $y = 5^x$. First, set the **WINDOW** so that $Xmin = -3$, $Xmax = 3$, $Ymin = -1$, and $Ymax = 10$.

ENTER: Y= (5 ÷ 2) ^ X,T,Θ,*n* ENTER

3 ^ X,T,Θ,*n* ENTER

5 ^ X,T,Θ,*n* GRAPH

DISPLAY:



If we press TRACE, the calculator will display the values $x = 0$, $y = 1$ for the first graph. By pressing ▼, the calculator will display this same set of values for the second function and again for the third. The point (0, 1) is a point on every function of the form $y = b^x$. For each function, as x decreases through negative values, the values of y get smaller and smaller but are always positive. We say that as x approaches $-\infty$, y approaches 0. The x-axis or the line $y = 0$ is a **horizontal asymptote**.

EXAMPLE 1

a. Sketch the graph of $y = \left(\frac{3}{2}\right)^x$.

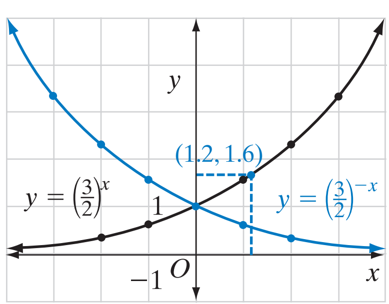
b. From the graph, estimate the value of $\left(\frac{3}{2}\right)^{1.2}$, the value of y when $x = 1.2$.

c. Sketch the graph of the image of the graph of $y = \left(\frac{3}{2}\right)^x$ under a reflection in the y-axis.

d. Write an equation of the graph drawn in part **c**.

Solution **a.**

x	y
−2	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
−1	$\left(\frac{3}{2}\right)^{-1} = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$
0	$\left(\frac{3}{2}\right)^0 = 1$
1	$\left(\frac{3}{2}\right)^1 = \frac{3}{2}$
2	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$
3	$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$



b. Locate the point 1.2 on the x-axis. Draw a vertical line to the graph. From that point on the graph draw a horizontal line to the y-axis. Read the approximate value of y from the graph. The value is approximately 1.6.

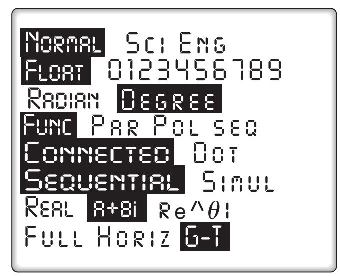
c. For each point (a, b) on the graph of $y = \left(\frac{3}{2}\right)^x$, locate the point $(-a, b)$ on the graph of the image.

d. The equation of the image is $y = \left(\frac{3}{2}\right)^{-x}$ or $y = \left(\frac{2}{3}\right)^x$.

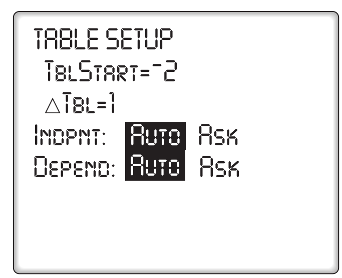
Answers **a.** Graph **b.** 1.6 **c.** Graph **d.** $y = \left(\frac{2}{3}\right)^x$

We can use the graphing calculator to find the values of the function used to graph $y = \left(\frac{3}{2}\right)^x$ in Example 1.

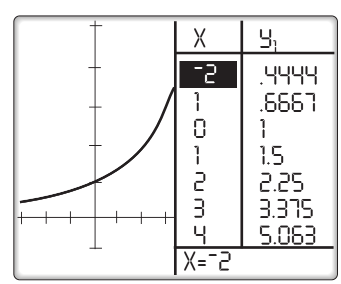
STEP 1. Enter the function into Y_1. Then change the graphing mode to split screen mode (G-T mode) by pressing **MODE** and selecting G-T. This will display the graph of the function alongside a table of values.



STEP 2. Press **2nd** **TBLSET** to enter the TABLE SETUP screen. Change TblStart to −2 and make sure that △Tbl is set to 1. This tells the calculator to start the table of values at −2 and increase each successive x-value by 1 unit.



STEP 3. Finally, press **GRAPH** to plot the graph and the table of values. Press **2nd** **TABLE** to use the table. Move through the table of values by pressing the up and down arrow keys.



By tracing values on the graph of $y = \left(\frac{3}{2}\right)^x$, we can again see that as x approaches $-\infty$, y approaches 0, and as x approaches ∞, y approaches ∞.

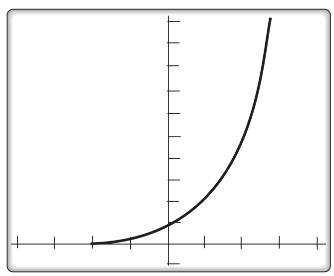
The Graph of $y = e^x$

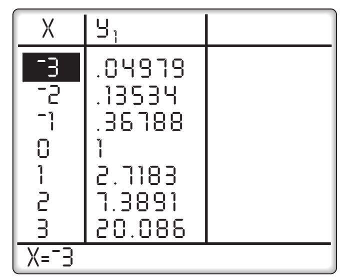
In Section 6-7, we introduced the irrational number e. For many real-world applications, the number e is a convenient choice to use as the base. This is why the number e is also called the **natural base**, and an exponential function with e as the base is called a **natural exponential function**. Recall that

$$e = 2.718281828\ldots$$

Note that e is an irrational number. The above pattern does *not* continue to repeat.

The figure on the left is a graph of the natural exponential function $y = e^x$. The figure on the right is a calculator screen of the same graph along with a table of selected values.





X	Y₁	
−3	.04979	
−2	.13534	
−1	.36788	
0	1	
1	2.7183	
2	7.3891	
3	20.086	

X=−3

Exercises

Writing About Mathematics

1. Explain why (0, 1) is a point on the graph of every function of the form $y = b^x$.

2. Explain why $y = b^x$ is *not* an exponential function for $b = 1$.

Developing Skills

In 3–6: **a.** Sketch the graph of each function. **b.** On the same set of axes, sketch the graph of the image of the reflection in the y-axis of the graph drawn in part **a**. **c.** Write an equation of the graph of the function drawn in part **b**.

3. $y = 4^x$ **4.** $y = 3^x$ **5.** $y = \left(\frac{7}{2}\right)^x$ **6.** $y = \left(\frac{3}{4}\right)^x$

7. a. Sketch the graph of $y = \left(\frac{5}{3}\right)^x$.

b. From the graph of $y = \left(\frac{5}{3}\right)^x$, estimate the value of y, to the nearest tenth, when $x = 2.2$.

c. From the graph of $y = \left(\frac{5}{3}\right)^x$, estimate the value of x, to the nearest tenth, when $y = 2.9$.

8. a. Sketch the graph of $f(x) = 2^x$.

b. Sketch the graph of the image of $f(x) = 2^x$ under a reflection in the x-axis.

c. Write an equation for the function whose graph was sketched in part **b**.

9. a. Sketch the graph of $f(x) = 1.2^x$.

b. Sketch the graph of the image of $f(x) = 1.2^x$ under a reflection in the x-axis.

c. Write an equation for the function whose graph was sketched in part **b**.

10. a. Make a table of values for e^x for integral values of x from -2 to 3.

b. Sketch the graph of $f(x) = e^x$ by plotting points and joining them with a smooth curve:

c. From the graph, estimate the value of $e^{\frac{1}{2}}$ and compare your answer to the value given by a calculator.

Applying Skills

11. The population of the United States can be modeled by the function $p(x) = 80.21e^{0.131x}$ where x is the number of decades (ten year periods) since 1900 and $p(x)$ is the population in millions.

a. Graph $p(x)$ over the interval $0 \leq x \leq 15$.

b. If the population of the United States continues to grow at this rate, predict the population in the years 2010 and 2020.

12. In 1986, the worst nuclear power plant accident in history occurred in the Chernobyl Nuclear Power Plant located in the Ukraine. On April 26, one of the reactors exploded, releasing large amounts of radioactive isotopes into the atmosphere. The amount of plutonium present after t years can be modeled by the function:

$$y = Pe^{-0.0000288t}$$

where P represents the amount of plutonium that is released.

a. Graph this function over the interval $0 \leq t \leq 100{,}000$ and $P = 10$ grams.

b. If 10 grams of the isotope plutonium-239 were released into the air, to the nearest hundredth, how many grams will be left after 10 years? After 100 years?

c. Using the graph, approximate how long it will take for the 10 grams of plutonium-239 to decay to 1 gram.

13. a. Graph the functions $y = x^4$ and $y = 4^x$ on a graphing calculator using the following viewing windows:

(1) $Xmin = 0, Xmax = 3, Ymin = 0, Ymax = 50$

(2) $Xmin = 0, Xmax = 5, Ymin = 0, Ymax = 500$

(3) $Xmin = 0, Xmax = 5, Ymin = 0, Ymax = 1{,}000$

b. How many points of intersection can you find? Find the coordinates of these intersection points to the nearest tenth.

c. Which function grows more rapidly for increasing values of x?

7-5 SOLVING EQUATIONS INVOLVING EXPONENTS

We know that to raise a power to a power, we multiply exponents. Therefore, for positive values of x and non-zero integer values of a:

$$(x^a)^{\frac{1}{a}} = x^{a\left(\frac{1}{a}\right)} = x^1 = x \qquad \left(x^{\frac{1}{a}}\right)^a = x^{\frac{1}{a}(a)} = x^1 = x$$

We can use this relationship to solve for x in an equation such as $x^{\frac{2}{3}} = 25$. To solve for x, we need to raise $x^{\frac{2}{3}}$ to the power that is the reciprocal of the exponent $\frac{2}{3}$. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

$$\begin{aligned} x^{\frac{2}{3}} &= 25 \\ \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} &= 25^{\frac{3}{2}} \\ x^1 &= 25^{\frac{3}{2}} \\ x &= 25^{\frac{3}{2}} \end{aligned}$$

Note that $25^{\frac{3}{2}}$ means $\left(25^{\frac{1}{2}}\right)^3$, that is, the cube of the square root of 25.

$$x = \left(\sqrt{25}\right)^3 = 5^3 = 125$$

EXAMPLE 1

Solve each equation and check: **a.** $2a^{-3} - 1 = 15$ **b.** $2\sqrt[3]{x^5} + 1 = 487$

Solution

How to Proceed

How to Proceed	a.	b.
(1) Write the equation with only the variable term on one side of the equation:	**a.** $2a^{-3} - 1 = 15$ $2a^{-3} = 16$	**b.** $2\sqrt[3]{x^5} + 1 = 487$ $2\sqrt[3]{x^5} = 487$
(2) Divide both sides of the equation by the coefficient of the variable term:	$a^{-3} = 8$	$x^{\frac{5}{3}} = 243$
(3) Raise both sides of the equation to the power that is the reciprocal of the exponent of the variable:	$(a^{-3})^{-\frac{1}{3}} = 8^{-\frac{1}{3}}$ $a = 8^{-\frac{1}{3}}$	$\left(x^{\frac{5}{3}}\right)^{\frac{3}{5}} = 243^{\frac{3}{5}}$ $x = 243^{\frac{3}{5}}$
(4) Simplify the right side of the equation:	$a = \frac{1}{8^{\frac{1}{3}}}$ $= \frac{1}{\sqrt[3]{8}}$ $= \frac{1}{2}$	$x = 243^{\frac{3}{5}}$ $= \left(\sqrt[5]{243}\right)^3$ $= 3^3$ $= 27$

(5) *Check* the solution:

a.

$$2a^{-3} - 1 = 15$$
$$2\left(\tfrac{1}{2}\right)^{-3} - 1 \stackrel{?}{=} 15$$
$$2\,(2)^3 - 1 \stackrel{?}{=} 15$$
$$2(8) - 1 \stackrel{?}{=} 15$$
$$16 - 1 \stackrel{?}{=} 15$$
$$15 = 15 \checkmark$$

b.

$$2\sqrt[3]{x^5} + 1 = 487$$
$$2\sqrt[3]{27^5} + 1 \stackrel{?}{=} 487$$
$$2\sqrt[3]{(3^3)^5} + 1 \stackrel{?}{=} 487$$
$$2\sqrt[3]{3^{15}} + 1 \stackrel{?}{=} 486$$
$$2(3^5) + 1 \stackrel{?}{=} 487$$
$$487 = 487 \checkmark$$

Answers **a.** $a = \frac{1}{2}$ **b.** $x = 27$

Exercises

Writing About Mathematics

1. Ethan said that to solve the equation $(x + 3)^{\frac{1}{2}} = 5$, the first step should be to square both sides of the equation. Do you agree with Ethan? Explain why or why not.

2. Chloe changed the equation $a^{-2} = 36$ to the equation $\frac{1}{a^2} = \frac{1}{36}$ and then took the square root of each side. Will Chloe's solution be correct? Explain why or why not.

Developing Skills

In 3–17 solve each equation and check.

3. $x^{\frac{1}{3}} = 4$ **4.** $a^{\frac{1}{5}} = 2$ **5.** $x^{\frac{2}{5}} = 9$

6. $b^{\frac{3}{2}} = 8$ **7.** $x^{-2} = 9$ **8.** $b^{-5} = \frac{1}{32}$

9. $2y^{-1} = 12$ **10.** $9a^{-\frac{3}{4}} = \frac{1}{3}$ **11.** $5x^{\frac{3}{4}} = 40$

12. $5x^{\frac{1}{2}} + 7 = 22$ **13.** $14 - 4b^{\frac{1}{3}} = 2$ **14.** $(2x)^{\frac{1}{2}} + 3 = 15$

15. $3a^3 = 81$ **16.** $x^5 = 3{,}125$ **17.** $z^{\frac{1}{2}} = \sqrt{81}$

In 18–23, solve for the variable in each equation. Express the solution to the nearest hundredth.

18. $x^{-3} = 24$ **19.** $y^{\frac{2}{3}} = 6$ **20.** $a^{-\frac{3}{4}} = 0.85$

21. $3z^3 + 2 = 27$ **22.** $5 + b^5 = 56$ **23.** $(3w)^9 + 2 = 81$

24. Solve for x and check: $\frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}} = 10$. Use the rule for the division of powers with like bases to simplify the left side of the equation.

Applying Skills

25. Show that if the area of one face of a cube is B, the volume of the cube is $B^{\frac{3}{2}}$.

26. If the area of one face of a cube is B and the volume of the cube is V, express B in terms of V.

7-6 SOLVING EXPONENTIAL EQUATIONS

Solving Exponential Equations With the Same Base

An **exponential equation** is an equation that contains a power with a variable exponent. For example, $2^{2x} = 8$ and $5^{x-1} = 0.04$ are exponential equations.

An exponential function $y = b^x$ is a one-to-one function since it is increasing for $b > 1$ and decreasing for $0 < b < 1$. Let $y_1 = b^{x_1}$ and $y_2 = b^{x_2}$. If $y_1 = y_2$, then $b^{x_1} = b^{x_2}$ and $x_1 = x_2$.

► **In general, if $b^p = b^q$, then $p = q$.**

We can use this fact to solve exponential equations that have the same base.

EXAMPLE 1

Solve and check: $3^x = 3^{2x-2}$

Solution Since the bases are equal, the exponents must be equal.

$$3^x = 3^{2x-2}$$
$$x = 2x - 2$$
$$-x = -2$$
$$x = 2$$

Check

$$3^x = 3^{2x-2}$$
$$3^2 \stackrel{?}{=} 3^{2(2)-2}$$
$$3^2 = 3^2 \checkmark$$

Answer $x = 2$

Solving Exponential Equations With Different Bases

How do we solve exponential equations such as $2^{2x} = 8$ or $5^{x-1} = 0.04$? One approach is, if possible, to write each term as a power of the same base. For example:

$$2^{2x} = 8$$
$$2^{2x} = 2^3$$
$$2x = 3$$
$$x = \frac{3}{2}$$

$$5^{x-1} = 0.04$$
$$5^{x-1} = \frac{4}{100}$$
$$5^{x-1} = \frac{1}{25}$$
$$5^{x-1} = \frac{1}{5^2}$$
$$5^{x-1} = 5^{-2}$$
$$x - 1 = -2$$
$$x = -1$$

EXAMPLE 2

Solve and check: $4^a = 8^{a+1}$

Solution The bases, 4 and 8, can each be written as a power of 2: $4 = 2^2$, $8 = 2^3$.

$$4^a = 8^{a+1}$$
$$(2^2)^a = (2^3)^{a+1}$$
$$2^{2a} = 2^{3a+3}$$
$$2a = 3a + 3$$
$$-a = 3$$
$$a = -3$$

Check

$$4^a = 8^{a+1}$$
$$4^{-3} \stackrel{?}{=} 8^{-3+1}$$
$$4^{-3} \stackrel{?}{=} 8^{-2}$$
$$\frac{1}{4^3} \stackrel{?}{=} \frac{1}{8^2}$$
$$\frac{1}{64} = \frac{1}{64} \checkmark$$

Answer $a = -3$

EXAMPLE 3

Solve and check: $3 + 7^{x-1} = 10$

Solution Add -3 to each side of the equation to isolate the power.

$$3 + 7^{x-1} = 10$$
$$7^{x-1} = 7$$
$$x - 1 = 1$$
$$x = 2$$

Check

$$3 + 7^{x-1} = 10$$
$$3 + 7^{2-1} \stackrel{?}{=} 10$$
$$3 + 7^1 \stackrel{?}{=} 10$$
$$10 = 10 \checkmark$$

Answer $x = 2$

Exercises

Writing About Mathematics

1. What value of a makes the equation $6^a = 1$ true? Justify your answer.

2. Explain why the equation $3^a = 5^{a-1}$ cannot be solved using the procedure used in this section.

Developing Skills

In 3–14, write each number as a power.

3. 9 **4.** 27 **5.** 25 **6.** 49

7. 1,000 **8.** 32 **9.** $\frac{1}{8}$ **10.** $\frac{1}{216}$

11. 0.001 **12.** 0.125 **13.** 0.81 **14.** 0.16

In 15–38, solve each equation and check.

15. $2^x = 16$
16. $3^x = 27$
17. $5^x = \frac{1}{5}$
18. $7^x = \frac{1}{49}$
19. $4^{x+2} = 4^{2x}$
20. $3^{x+1} = 3^{2x+3}$
21. $6^{3x} = 6^{x-1}$
22. $3^{x+2} = 9^x$
23. $25^x = 5^{x+3}$
24. $49^x = 7^{3x+1}$
25. $2^{2x+1} = 16^x$
26. $9^{x-1} = 27^x$
27. $100^x = 1{,}000^{x-1}$
28. $125^{x-1} = 25^x$
29. $6^{2-x} = \left(\frac{1}{36}\right)^2$
30. $\left(\frac{1}{4}\right)^x = 8^{1-x}$
31. $\left(\frac{1}{3}\right)^x = 9^{1-x}$
32. $(0.01)^{2x} = 100^{2-x}$
33. $(0.25)^{x-2} = 4^x$
34. $5^{x-1} = (0.04)^{2x}$
35. $4^x + 7 = 15$
36. $5 + 7^x = 6$
37. $e^{2x+2} = e^{x-1}$
38. $3^{x^2+2} = 3^6$

7-7 APPLICATIONS OF EXPONENTIAL FUNCTIONS

There are many situations in which an initial value A_0 is increased or decreased by adding or subtracting a percentage of A_0. For example, the value of an investment is the amount of money invested plus the interest, a percentage of the investment.

Let A_n be the value of the investment after n years if A_0 dollars are invested at rate r per year.

- After 1 year: $A_1 = A_0 + A_0r = A_0(1 + r)$

  $A_1 = A_0(1 + r)$
- After 2 years: $A_2 = A_1 + A_1r = A_0(1 + r) + A_0(1 + r)r$
  $= A_0(1 + r)(1 + r) = A_0(1 + r)^2$

  $A_2 = A_0(1 + r)^2$
- After 3 years: $A_3 = A_2 + A_2r = A_0(1 + r)^2 + A_0(1 + r)^2r$
  $= A_0(1 + r)^2(1 + r) = A_0(1 + r)^3$

  $A_3 = A_0(1 + r)^3$
- After 4 years: $A_4 = A_3 + A_3r = A_0(1 + r)^3 + A_0(1 + r)^3r$
  $= A_0(1 + r)^3(1 + r) = A_0(1 + r)^4$

  $A_4 = A_0(1 + r)^4$

We see that a pattern has been established. The values at the end of each year form a geometric sequence with the common ratio $(1 + r)$.

- After t years:

$$A_t = A_{t-1} + A_{t-1}r = A_0(1 + r)^{t-1} + A_0(1 + r)^{t-1}r = A_0(1 + r)^{t-1}(1 + r) = A_0(1 + r)^t$$

or

$$A = A_0(1 + r)^t$$

A calculator can be used to display the value of an investment year by year. The value of the investment after each year is $(1 + r)$ times the value of the investment from the previous year. If \$100 is invested at a yearly rate of 5%, the value of the investment at the end of each of the first 5 years can be found by multiplying the value from the previous year by $(1 + 0.05)$ or 1.05.

ENTER: 100 [×] 1.05 [ENTER]
[×] 1.05 [ENTER]
[×] 1.05 [ENTER]
[×] 1.05 [ENTER]
[×] 1.05 [ENTER]

DISPLAY:

```
100*1.05
                 105
Ans*1.05
              110.25
            115.7625
          121.550625
         127.6281563
```

The value of the investment for each of the first five years is \$105.00, \$110.25, \$115.76, \$121.55, and \$127.63.

EXAMPLE 1

Amanda won \$10,000 and decided to use it as a "vacation fund." Each summer, she withdraws an amount of money that reduces the value of the fund by 7.5% from the previous summer. How much will the fund be worth after the tenth withdrawal?

Solution Use the formula $A = A_0(1 + r)^t$ with $A_0 = 10{,}000$, $r = -0.075$ and $t = 10$.

$$A = 10{,}000(1 - 0.075)^{10} = 10{,}000(0.925)^{10}$$

ENTER: 10000 [×] 0.925 [^] 10 [ENTER]

DISPLAY:

```
10000*0.925^10
           4585.823414
```

Answer \$4,585.82

Calculator variables can be used to quickly evaluate an exponential model for different time periods. For instance, to find the value of the fund from Example 1 after the second and fifth withdrawals:

STEP 1. Let $Y_1 = 10{,}000(0.925)^x$.

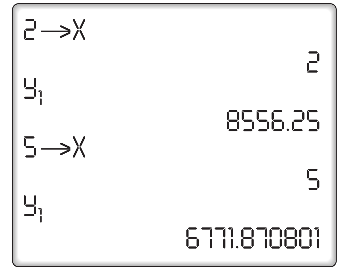
STEP 2. Exit the Y= screen and let $X = 2$.

ENTER: 2 **STO▶** **X,T,Θ,*n*** **ENTER**

STEP 3. Find the value of Y_1 when $X = 2$.

ENTER: **VARS** **▶** **ENTER** **ENTER** **ENTER**

STEP 4. Repeat steps 2 and 3 to find the value of Y_1 when $X = 5$.



The value of the fund after the second withdrawal is \$8,556.25 and after the fifth withdrawal, \$6,771.87.

Other Types of Compounding Periods

The exponential function $A = A_0(1 + r)^t$, when applied to investments, represents interest *compounded annually* since interest is added once at the end of each year. However, this formula can be applied to any increase that takes place at a fixed rate for any specified period of time. For example, if an annual interest rate of 5% is paid monthly, then interest of $\frac{5}{12}$% is paid 12 times in one year. Or if an annual interest rate of 5% is paid daily, then interest of $\frac{5}{365}$% is paid 365 times in one year. Compare the value of an investment of \$100 for these three cases.

Paid yearly for 1 year: $A = 100(1 + 0.05)^1 = 105$ or \$105.00

Paid monthly for 1 year: $A = 100\left(1 + \frac{0.05}{12}\right)^{12} \approx 105.1161898$ or \$105.12

Paid daily for 1 year: $A = 100\left(1 + \frac{0.05}{365}\right)^{365} \approx 105.1267496$ or \$105.13

In general, if interest is compounded n times in one time period and the number of time periods is t, then:

$$\mathbf{A = A_0\left(1 + \frac{r}{n}\right)^{nt}}$$

Note that as the number of times that the interest is compounded increases, the value of the investment for a fixed period of time increases.

What happens as we let n, the number of compoundings, increase without limit? Let $\frac{r}{n} = \frac{1}{k}$. Then $n = rk$.

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = A_0\left(1 + \frac{1}{k}\right)^{rkt} \qquad \text{Substitute } \frac{r}{n} = \frac{1}{k} \text{ and } n = rk.$$

$$A = A_0\left[\left(1 + \frac{1}{k}\right)^{k}\right]^{rt}$$

As n approaches infinity, k approaches infinity. It can be shown that $\left(1 + \frac{1}{k}\right)^k$ approaches e. The table on the right shows values of $\left(1 + \frac{1}{k}\right)^k$ rounded to eight decimal places for different values of k. Recall that e is an irrational number that is the sum of an infinite series:

k	$\left(1 + \frac{1}{k}\right)^k$
1	2
10	2.59374246
100	2.70481383
1,000	2.71692393
100,000	2.71826823
1,000,000	2.71828047

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718281828\ldots$$

Therefore, for very large values of n:

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt} = A_0\left(\left(1 + \frac{1}{k}\right)^k\right)^{rt} = A_0e^{rt}$$

This formula can be applied to any change that takes place continuously at a fixed rate. Large populations of people or of animals can be said to increase continuously. If this happens at a fixed rate per year, then the size of the population in the future can be predicted.

EXAMPLE 2

In a state park, the deer population was estimated to be 2,000 and increasing continuously at a rate of 4% per year. If the increase continues at that rate, what is the expected deer population in 10 years?

Solution Use the formula $A = A_0e^{rt}$. Let $A_0 = 2{,}000$, $r = 0.04$, and $t = 10$.

$$A = 2{,}000e^{0.04(10)} = 2{,}000e^{0.4}$$

ENTER: 2000 [×] [2nd] [e^x] 0.4 [)] [ENTER]

DISPLAY:

```
2000*e^(0.4)
         2983.649395
```

Answer The expected population is 2,984 deer.

Note: Since the answer in Example 2 is an approximation, it is probably more realistic to say that the expected population is 3,000 deer.

The formula $A = A_0e^{rt}$ can also be used to study radioactive decay. Decay takes place continuously. The decay constant for a radioactive substance is negative to indicate a decrease.

EXAMPLE 3

The decay constant of radium is -0.0004 per year. How many grams will remain of a 50-gram sample of radium after 20 years?

Solution Since change takes place continuously, use the formula $A = A_0e^{rt}$. Let $A_0 = 50$, $r = -0.0004$ and $t = 20$.

$$A = 50e^{-0.0004(20)} = 50e^{-0.008}$$

ENTER: 50 [×] [2nd] [e^x] -0.008 [)] [ENTER]

DISPLAY:

```
50*e^(-0.008)
       49.60159574
```

Answer Approximately 49.6 grams

Exercises

Writing About Mathematics

1. Show that the formula $A = A_0(1 + r)^n$ is equivalent to $A = A_0(2)^n$ when $r = 100\%$.
2. Explain why, if an investment is earning interest at a rate of 5% per year, the investment is worth more if the interest is compounded daily rather than if it is compounded yearly.

Developing Skills

In 3–10, find the value of x to the nearest hundredth.

3. $x = e^2$ **4.** $x = e^{1.5}$ **5.** $x = e^{-1}$ **6.** $xe^3 = e^4$

7. $12x = e$ **8.** $\frac{x}{e^3} = e^{-2}$ **9.** $x = e^3e^5$ **10.** $x = e^3 + e^5$

In 11–16, use the formula $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ to find the missing variable to the nearest hundredth.

11. $A_0 = 50, r = 2\%, n = 12, t = 1$

12. $A = 400, r = 5\%, n = 4, t = 3$

13. $A = 100, A_0 = 25, n = 1, t = 2$

14. $A = 25, A_0 = 200, r = -50\%, n = 1$

15. $A = 250, A_0 = 10, n = 3, t = 1$

16. $A = 6, A_0 = 36, n = 1, t = 4$

Applying Skills

17. A bank offers certificates of deposit with variable compounding periods.

a. Joe invested \$1,000 at 6% per year compounded yearly. Find the values of Joe's investment at the end of each of the first five years.

b. Sue invested \$1,000 at 6% per year compounded monthly. Find the values of Sue's investment at the end of each of the first five years.

c. Who had more money after the end of the fifth year?

d. The *annual percentage yield* (APY) is the amount an investment actually increases during one year. Find the APY for Joe and Sue's certificates of deposit. Is the APY of each investment equal to 6%?

18. a. When Kyle was born, his grandparents invested \$5,000 in a college fund that paid 4% per year, compounded yearly. What was the value of this investment when Kyle was ready for college at age 18? (Note that $r = 0.04$.)

b. If Kyle's grandparents had invested the \$5,000 in a fund that paid 4% compounded continuously, what would have been the value of the fund after 18 years?

19. A trust fund of \$2.5 million was donated to a charitable organization. Once each year the organization spends 2% of the value of the fund so that the fund decreases by 2%. What will be the value of the fund after 25 years?

20. The decay constant of a radioactive element is -0.533 per minute. If a sample of the element weighs 50 grams, what will be its weight after 2 minutes?

21. The population of a small town decreased continually by 2% each year. If the population of the town is now 37,000, what will be the population 8 years from now if this trend continues?

22. A piece of property was valued at \$50,000 at the end of 1990. Property values in the city where this land is located increase by 10% each year. The value of the land increases continuously. What is the property worth at the end of 2010?

23. A sample of a radioactive substance decreases continually at a rate of -0.04. If the weight of the sample is now 40 grams, what will be its weight in 7 days?

24. The number of wolves in a wildlife preserve is estimated to have increased continually by 3% per year. If the population is now estimated at 5,400 wolves, how many were present 10 years ago?

25. The amount of a certain medicine present in the bloodstream decreases at a rate of 10% per hour.

a. Which is a better model to use for this scenario: $A = A_0(1 + r)^t$ or $A = A_0e^{rt}$? Explain your answer.

b. Using both models, find the amount of medicine in the bloodstream after 10.5 hours if the initial dose was 200 milligrams.

CHAPTER SUMMARY

The rules for operations with powers with like bases can be extended to include zero, negative, and fractional exponents. If $x > 0$ and $y > 0$:

Multiplication: $x^a \cdot x^b = x^{a+b}$

Division: $x^a \div x^b = x^{a-b}$ **or** $\frac{x^a}{x^b} = x^{a+b}$ $(x \neq 0)$

Raising to a Power: $(x^a)^b = x^{ab}$

Power of a Product: $(xy)^a = x^a \cdot y^a$

Power of a Quotient: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

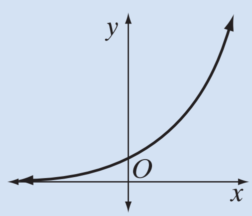
If $x > 0$ and n is a positive integer:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

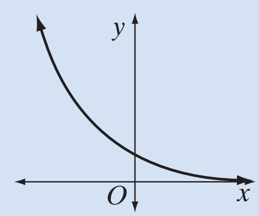
In general, for $x > 0$ and m and n positive integers:

$$\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m = x^{\frac{m}{n}} \qquad \frac{1}{\sqrt[n]{x^m}} = \frac{1}{\left(\sqrt[n]{x}\right)^m} = x^{-\frac{m}{n}}$$

The graph of the exponential function, $y = b^x$:



for $b > 1$



for $0 < b < 1$

To solve an equation of the form $x^{\frac{a}{b}} = c$, raise each side of the equation to the $\frac{b}{a}$ power so that the left member is x^1 or x.

An **exponential equation** is an equation that contains a power with a variable exponent. To solve an exponential equation, write each member with the same base and equate exponents.

The number e is also called the **natural base** and the function $y = e^x$ is called the **natural exponential function**.

If a quantity A_0 is increased or decreased by a rate r per interval of time, its value A after t intervals of time is $A = A_0(1 + r)^t$.

If a quantity A_0 is increased or decreased by a rate r per interval of time, compounded n times per interval, its value A after t intervals of time is:

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If the increase or decrease is continuous for t units of time, the formula becomes

$$A = A_0e^{rt}$$

VOCABULARY

7-4 Exponential growth • Exponential decay • Exponential function • Horizontal asymptote • Natural base • Natural exponential function

7-6 Exponential equation

REVIEW EXERCISES

In 1–12, evaluate each expression.

1. 3^0 **2.** 2^{-1} **3.** $8^{\frac{1}{3}}$ **4.** $5(3)^0$

5. $8(2)^3$ **6.** $5(0.1)^{-2}$ **7.** $12(9)^{\frac{1}{2}}$ **8.** $(2 \times 3)^{-2}$

9. $(9 \times 9)^{\frac{1}{4}}$ **10.** $100^{-\frac{1}{2}}$ **11.** $125^{\frac{2}{3}}$ **12.** $\left(12 \times \frac{4}{3}\right)^{-\frac{3}{2}}$

In 13–16, evaluate each function for the given value. Be sure to show your work.

13. $g(x) = x^{\frac{1}{4}}x^{\frac{1}{4}}$; $g(4)$

14. $g(x) = \left(\frac{1}{1 + \frac{1}{x^{-3}}}\right)$; $g(3)$

15. $g(x) = 2^x + \left(\frac{1}{2^x}\right)$; $g(4)$

16. $g(x) = \left(\frac{x^5}{x^7}\right)^{-2}$; $g(10)$

In 17–20, write each expression with only positive exponents and express the answer in simplest form. The variables are not equal to zero.

17. $(27x)^3 \cdot 3(x)^{-1}$

18. $\left(\frac{a^3bc}{abc^3}\right)^{-3}$

19. $\left(\frac{x^3}{b^{-2}}\right)^{-\frac{1}{5}}$

20. $\left(2x^2y^{-2}z^{\frac{3}{2}}\right)(3x^{-3}y^3z^{-1})$

In 21–24, write each radical expression as a power with positive exponents and express the answer in simplest form. The variables are positive numbers.

21. $\frac{3}{\sqrt{25x}}$

22. $\sqrt[3]{\frac{81y^3}{3y^5}}$

23. $\sqrt[12]{64a^7b^{20}}$

24. $\sqrt[6]{32x^8y^3}$

In 25–28, write each power as a radical expression in simplest form. The variables are positive numbers.

25. $(2y)^{\frac{5}{2}}$

26. $(16x^8y)^{\frac{1}{2}}$

27. $\left(\frac{(a + 2)}{2^4}\right)^{\frac{3}{4}}$

28. $\left(\frac{a^{12}b^9c}{a^6b^3}\right)^{\frac{1}{4}}$

29. **a.** Sketch the graph of $y = 1.25^x$.

b. On the same set of axes, sketch the graph of $y = 0.80^x$.

c. Under what transformation is the graph of $y = 0.80^x$ the image of $y = 1.25^x$?

In 30–41, solve and check each equation.

30. $x^{-2} = 36$

31. $a^{\frac{1}{3}} = 4$

32. $b^{-\frac{1}{2}} = \frac{2}{3}$

33. $y^{-3} + 6 = 14$

34. $a^{\frac{2}{3}} + 7 = 9$

35. $4^x + 2 = 10$

36. $2(5)^{-x} = 50$

37. $7^x = 7^{2x-2}$

38. $2^{x+2} = 8^{x-2}$

39. $0.10^{-x} = 10^{2x+1}$

40. $3^{x+1} = 27^x$

41. $2 = 0.5^x$

42. Find the interest that has accrued on an investment of $500 if interest of 4% per year is compounded quarterly for a year.

43. The label on a prescription bottle directs the patient to take the medicine twice each day. The effective ingredient of the medicine decreases continuously at a rate of 25% per hour. If a dose of medicine containing 1 milligram of the effective ingredient is taken, how much is still present 12 hours later when the second dose is taken?

44. A sum of money that was invested at a fixed rate of interest doubled in value in 15 years. The interest was compounded yearly. Find the rate of interest to the nearest tenth of a percent.

45. A company records the value of a machine used for production at $25,000. As the machine ages, its value depreciates, that is, decreases in value. If the depreciation is estimated to be 20% of the value of the machine at the end of each year, what is the expected value of the machine after 6 years?

46. Determine the common solution of the system of equations:

$$15y = 27^x$$
$$5y = 3^x$$

Exploration

Prove that for all a:

a. $\frac{3^a - 3^{a-2}}{3^{a-1} + 3^a} = \frac{2}{3}$

b. $\frac{4^{a+1} + 4}{2^{a+5} \cdot 2^{a-1} + 16} = \frac{1}{4}$

CUMULATIVE REVIEW — CHAPTERS 1–7

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The expression $2x^3(4 - 3x) - 5x(x - 2)$ is equal to

(1) $6x^4 - 8x^3 + 5x^2 - 10x$

(2) $-6x^4 + 8x^3 - 5x^2 + 10x$

(3) $6x^4 + 8x^3 - 5x^2 + 10x$

(4) $-6x^4 + 8x^3 - 5x^2 - 10x$

2. If the domain is the set of integers, the solution set of $x^2 - 3x - 4 < 0$ is
(1) $\{-1, 0, 1, 2, 3, 4\}$
(2) $\{0, 1, 2, 3\}$
(3) $\{\ldots, -3, -2, -1, 4, 5, 6, \ldots\}$
(4) $\{\ldots, -4, -3, -2, 5, 6, 7, \ldots\}$

3. The fraction $\dfrac{\frac{1}{3} - 1}{\frac{1}{9} - 3}$ is equal to
(1) -1
(2) 1
(3) $\frac{3}{13}$
(4) $\frac{52}{27}$

4. An equation whose roots are -3 and 5 is
(1) $x^2 + 2x - 15 = 0$
(2) $x^2 - 2x - 15 = 0$
(3) $x^2 + 2x + 15 = 0$
(4) $x^2 - 2x + 15 = 0$

5. In simplest form, $\sqrt{12} + \sqrt{9} + \sqrt{27}$ is
(1) $5\sqrt{3} + 3$
(2) $6\sqrt{3}$
(3) $4\sqrt{3}$
(4) $\sqrt{48}$

6. The expression $\sqrt{-2}(\sqrt{-18}) + \sqrt{-25}$ is equal to
(1) 1
(2) $-6 + 5i$
(3) $6 + 5i$
(4) $11i$

7. The discriminant of a quadratic equation is 35. The roots are
(1) unequal rational numbers.
(2) unequal irrational numbers.
(3) equal rational numbers.
(4) imaginary numbers.

8. In the sequence $100, 10, 1, \ldots, a_{20}$ is equal to
(1) 10^{-20}
(2) 10^{-19}
(3) 10^{-18}
(4) 10^{-17}

9. The factors of $x^3 - 4x^2 - x + 4$ are
(1) $(x - 4)(x - 1)(x + 1)$
(2) $(x - 2)(x - 1)(x + 1)(x + 2)$
(3) $(x + 4)(x - 1)(x + 1)$
(4) $(4 - x)(x - 1)(x + 1)$

10. The solution set of the equation $6 - \sqrt{7 - x} = 8$ is
(1) $\{3\}$
(2) $\{-3\}$
(3) $\{-3, 3\}$
(4) $\varnothing$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Write an expression for the nth term of the series $1 + 3 + 5 + 7 + \cdots$.

12. The roots of a quadratic equation are 2 and $\frac{5}{2}$. Write the equation.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Express $\frac{3 + \sqrt{-4}}{1 - \sqrt{-4}}$ in $a + bi$ form.

14. Find the solution set of the equation $1 + 27^{x+1} = 82$.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. What is the solution set of the inequality $x^2 - 7x - 12 < 0$?

16. A circle with center at $(2, 0)$ and radius 4 is intersected by a line whose slope is 1 and whose y-intercept is 2.

a. Write the equation of the circle in center-radius form.

b. Write the equation of the circle in standard form.

c. Write the equation of the line.

d. Find the coordinates of the points at which the line intersects the circle.